

# On the well-posedness of the Cauchy problem for the generalized Korteweg-de Vries-Burgers equation \*

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**Abstract** Considered is the generalized Korteweg-de Vries-Burgers equation

$$u_t + u_{xxx} + uu_x + |D_x|^{2\alpha} u = 0, \quad t \in \mathbb{R}^+, x \in \mathbb{R},$$

with  $0 \leq \alpha \leq 1$ . We prove a sharp results on the associated Cauchy problem in the Sobolev space  $H^s(\mathbb{R})$ . For  $s > -\min\{\frac{3+2\alpha}{4}, 1\}$  we give the well-posedness of solutions of the Cauchy problem, while for  $\frac{1}{2} \leq \alpha \leq 1$  and for  $s < -\min\{\frac{3+2\alpha}{4}, 1\}$  we show some ill-posedness issues.

**Key words** Korteweg-de Vries-Burgers equation; well-posedness; existence.

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## 1 Introduction and statement of the result

In this paper we consider the cauchy problem associated with the generalized Korteweg-de Vries-Burgers equation

$$\begin{cases} u_t + u_{xxx} + uu_x + |D_x|^{2\alpha} u = 0, & t \in \mathbb{R}^+, x \in \mathbb{R} \\ u(0) = \varphi(x), \end{cases} \quad (1.1)$$

where,  $0 \leq \alpha \leq 1$ ,  $|D_x|^{2\alpha}$  is the Fourier multiplier associated with the symbol  $|\xi|^{2\alpha}$ .

Equation (1.1) has been derived as a model for the propagation of weakly nonlinear dispersive long waves in some physical contexts when dissipative effects occur (see [1]). The long time asymptotic behavior of its solutions has been studied in numerous papers (see [2] and references therein ).

When  $\alpha = 0$ , (1.1) is the Korteweg-de Vries equation. The best known results on the Cauchy problem for the Korteweg-de Vries equation have been derived by Kenig, Ponce and Vega (see [3], [4]). They proved that the Cauchy problem for the KdV equation is locally well-posed in  $H^s(\mathbb{R})$  for  $s > -\frac{3}{4}$ , and that the flow-map for the KdV equation is not locally uniformly continuous in  $H^s(\mathbb{R})$  for  $s < -\frac{3}{4}$ . For the Cauchy problem of the dissipative Burgers equation

$$u_t - u_{xx} + uu_x = 0,$$

it is known that the local well-posedness in  $H^s(\mathbb{R})$  holds for  $s \geq -\frac{1}{2}$  (see [9]), and some non-uniqueness phenomena occur for  $s < -\frac{1}{2}$  (see [6]). When  $\alpha = 1$ , (1.1) is the Korteweg-de Vries-Burgers equation. Molinet and Ribaud in [7] proved that the Korteweg-de Vries-Burgers equation is globally well-posed in  $H^s(\mathbb{R})$  for  $s > -1$  and ill-posed in  $H^s(\mathbb{R})$  for  $s < -1$ . They proved that the Cauchy problem (1.1) associated with  $0 \leq \alpha \leq 1$  is ill-posed in the homogenous Sobolev space  $\dot{H}^s(\mathbb{R})$  for  $s < \frac{\alpha-3}{2(2-\alpha)}$ , and conjectured that the well-posedness in  $H^s(\mathbb{R})$  for  $s > \frac{\alpha-3}{2(2-\alpha)}$  could be

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proved. The aim of this paper is to answer this open problem. We prove that (1.1) is well-posed in the Sobolev space  $H^s(\mathbb{R})$  for  $s > -\min\{\frac{3+2\alpha}{4}, 1\}$ . Note that  $-\min\{\frac{3+2\alpha}{4}, 1\} < \frac{\alpha-3}{2(2-\alpha)}$  for  $0 < \alpha < 1$ .

Let  $\langle \cdot \rangle = (1 + |\cdot|^2)^{\frac{1}{2}}$ . We define

$$X_\alpha^{b,s} = \{u \in \mathcal{S}'(\mathbb{R}^2) : \|u\|_{X_\alpha^{b,s}} < +\infty\},$$

$$X_{\alpha,T}^{b,s} = \{u : \exists v \in X_\alpha^{b,s} \text{ satisfying } u = v \text{ in } \mathbb{R} \times [0, T]\},$$

with

$$\|u\|_{X_\alpha^{b,s}} = \|\langle i(\tau - \xi^3) + |\xi|^{2\alpha} \rangle^b \langle \xi \rangle^s \hat{u}(\xi, \tau)\|_{L^2(\mathbb{R}^2)},$$

$$\|u\|_{X_{\alpha,T}^{b,s}} = \inf\{\|v\|_{X_\alpha^{b,s}} : v \in X_\alpha^{b,s} \text{ satisfying } u = v \text{ in } \mathbb{R} \times [0, T]\}.$$

Let  $H^s(\mathbb{R})$  be the usual Sobolev space. Our main result is

**Theorem 1.1** *Let  $\varphi \in H^s(\mathbb{R})$  with  $s > -\min\{\frac{3+2\alpha}{4}, 1\}$ . For any  $T > 0$ , there exists a unique solution  $u$  of (1.1) satisfying*

$$u \in Z_T = C([0, T], H^s(\mathbb{R})) \cap X_{\alpha,T}^{\frac{1}{2},s}.$$

Moreover the map  $\varphi \mapsto u$  is smooth from  $H^s(\mathbb{R})$  to  $Z_T$  and  $u$  belongs to  $C((0, +\infty), H^\infty(\mathbb{R}))$ .

**Remark** For  $s < \frac{\alpha-3}{2(2-\alpha)}$ , Molinet and Ribaud (see Remark 1 and Theorem 2 in [7]) proved that the flow-map

$$\varphi \mapsto u(t), t \in [0, T]$$

is not  $C^2$  differentiable at zero from the homogenous Sobolev  $\dot{H}^s(\mathbb{R})$  to  $C([0, T]; \dot{H}^s(\mathbb{R}))$ .

The result is optimal in the case  $\frac{1}{2} \leq \alpha \leq 1$ .

**Theorem 1.2** *Let  $\frac{1}{2} \leq \alpha \leq 1$  and  $s < -1$ . Then there does not exist  $T > 0$  such that the Cauchy problem (1.1) has a unique local solution  $u$  defined on the interval  $[0, T]$ , and such that the flow-map*

$$\varphi \mapsto u(t), t \in [0, T]$$

is  $C^2$  differentiable at zero from  $H^s(\mathbb{R})$  to  $C([0, T], H^s(\mathbb{R}))$ .

In this paper, we use  $A \lesssim B$  to denote the statement that  $A \leq CB$  for some large constant  $C$  which may vary from line to line, and similarly use  $A \ll B$  to denote the statement  $A \leq C^{-1}B$ . We use  $A \sim B$  to denote the statement that  $A \lesssim B \lesssim A$ . Any summations over capitalized variables such as  $N_j, L_j, H$  are presumed to be dyadic, i.e. these variables range over numbers of the form  $2^k$  for  $k \in \mathbb{Z}$  or for  $k \in \mathbb{N}$ . In addition to the usual notation  $\chi_E$  for characteristic functions, we define  $\chi_P$  for statements  $P$  to be 1 if  $P$  is true and 0 otherwise, e.g.  $\chi_{1 \leq |\xi| \leq 2}$ .

We adopt the following summation conventions. Any summation of the form  $L_{max} \sim \cdot$  is a sum over the three dyadic variables  $L_1, L_2, L_3 \gtrsim 1$ , thus for instance

$$\sum_{L_{max} \sim H} := \sum_{L_1, L_2, L_3 \gtrsim 1; L_{max} \sim H}.$$

Similarly, any summation of the form  $N_{max} \sim \cdot$  sum over the three dyadic variables  $N_1, N_2, N_3 > 0$ , thus for instance

$$\sum_{N_{max} \sim N_{med} \sim N} := \sum_{N_1, N_2, N_3 > 0; N_{max} \sim N_{med} \sim N}.$$

The rest of this paper is organized as follows. In section 2 we give some linear estimates. In section 3 we prove the crucial bilinear estimates and give the proof of Theorem 1.1. The ill-posedness is given in section 4.

## 2 Linear estimates

Let  $U(\cdot)$  be the free evolution of the KdV equation defined by  $U(t) = e^{itP(D_x)}$ , where  $P(D_x)$  is the Fourier multiplier with the symbol  $P(\xi) = \xi^3$ . Obviously  $U(\cdot)$  is a unitary group in  $H^s(\mathbb{R})$ ,  $s \in \mathbb{R}$ . Since  $\mathcal{F}(U(-t)u)(\tau, \xi) = \mathcal{F}(u)(\tau + \xi^3, \xi)$ , one can rewrite the norm of  $X_\alpha^{b,s}$  as

$$\|u\|_{X_\alpha^{b,s}} = \|\langle i\tau + |\xi|^{2\alpha} \rangle^b \langle \xi \rangle^s \mathcal{F}(U(-t)u)(\tau, \xi)\|_{L^2(\mathbb{R}^2)}.$$

Let  $W(\cdot)$  be the semigroup associated with the free evolution of (1.1) defined by

$$\mathcal{F}_x(W(t)\varphi)(\xi) = e^{it\xi^3 - t|\xi|^{2\alpha}} \hat{\varphi}(\xi), \quad \varphi \in \mathcal{S}'(\mathbb{R}), \quad t \geq 0,$$

and we extend  $W(\cdot)$  to a linear operator defined on the whole real axis by setting

$$\mathcal{F}_x(W(t)\varphi)(\xi) = e^{it\xi^3 - t|\xi|^{2\alpha}} \hat{\varphi}(\xi), \quad \varphi \in \mathcal{S}'(\mathbb{R}), \quad t \in \mathbb{R}.$$

Let  $\psi$  be a time cut-off function defined by

$$\psi \in C_0^\infty(\mathbb{R}), \quad \text{supp}\psi \subset [-2, 2], \quad \psi \equiv 1 \quad \text{on} \quad [-1, 1]$$

and let  $\psi_T(\cdot) = \psi(\cdot/T)$  for a given  $T > 0$ .

**Proposition 2.1** *For  $s \in \mathbb{R}$ , we have*

$$\|\psi(t)W(t)\varphi\|_{X_\alpha^{\frac{1}{2},s}} \lesssim \|\varphi\|_{H^s}, \quad \forall \varphi \in H^s(\mathbb{R}).$$

*Proof.* Set  $g_\xi = \psi(t)e^{-|t||\xi|^{2\alpha}}$ . For  $b \in \{0, \frac{1}{2}\}$  we have

$$\|g_\xi\|_{H_t^b} \leq \|\langle \tau \rangle^b \hat{\psi}\|_{L^1} \|e^{-|t||\xi|^{2\alpha}}\|_{L^2} + \|\hat{\psi}\|_{L^1} \|e^{-|t||\xi|^{2\alpha}}\|_{H_t^b}.$$

Since  $\psi \in C_0^\infty(\mathbb{R})$ ,  $\text{supp}\psi \subset [-2, 2]$ , we get  $\|\langle \tau \rangle^b \hat{\psi}\|_{L^1} \leq C$ . Note that

$$\|e^{-|t||\xi|^{2\alpha}}\|_{H_t^b} \sim (|\xi|^{2\alpha})^{b-\frac{1}{2}} \|e^{-|t|}\|_{H_t^b}.$$

We deduce for  $|\xi| \geq 1$

$$\|g_\xi\|_{H_t^b} \lesssim (|\xi|^{-\alpha} + |\xi|^{2\alpha b - \alpha}) \leq C |\xi|^{2\alpha(b-\frac{1}{2})}, \quad (2.1)$$

and for  $|\xi| \leq 1$ ,

$$\|g_\xi\|_{H_t^b} \leq \sum_{n=0}^{\infty} \frac{|\xi|^{2\alpha n}}{n!} \|\psi(t)t^n\|_{H_t^b} \leq \sum_{n=0}^{\infty} \frac{|\xi|^{2\alpha n}}{n!} \|\psi(t)t^n\|_{H_t^1} \lesssim 1. \quad (2.2)$$

A combination of (2.1) with (2.2) yields

$$\|g_\xi\|_{H_t^b} \lesssim \langle \xi \rangle^{\alpha(2b-1)}, \quad b = 0 \quad \text{or} \quad \frac{1}{2}. \quad (2.3)$$

By (2.3), we have

$$\begin{aligned} & \|\psi(t)W(t)\varphi\|_{X_\alpha^{\frac{1}{2},s}} \\ & \lesssim \left\| \langle \xi \rangle^s \hat{\varphi}(\xi) \|\langle \tau \rangle^{\frac{1}{2}} \mathcal{F}_t(\psi(t)e^{-|t||\xi|^{2\alpha}})(\tau)\|_{L_\tau^2} \right\|_{L_\xi^2} \\ & \quad + \left\| \langle \xi \rangle^{s+\alpha} \hat{\varphi}(\xi) \|\psi(t)e^{-|t||\xi|^{2\alpha}}\|_{L_t^2} \right\|_{L_\xi^2} \\ & \lesssim \left\| \langle \xi \rangle^s \hat{\varphi}(\xi) \|g_\xi(t)\|_{H_t^{\frac{1}{2}}} \right\|_{L_\xi^2} + \left\| \langle \xi \rangle^{s+\alpha} \hat{\varphi}(\xi) \|g_\xi(t)\|_{H_t^0} \right\|_{L_\xi^2} \\ & \lesssim \|\langle \xi \rangle^s \hat{\varphi}(\xi)\|_{L_\xi^2} + C \|\langle \xi \rangle^s \hat{\varphi}(\xi)\|_{L_\xi^2} \lesssim \|\varphi\|_{H^s}. \end{aligned}$$

□

The following proposition comes from Proposition 2 in [7] (we replace  $\xi$  by  $|\xi|^{2\alpha}$ ).

**Proposition 2.2** For  $\omega \in \mathcal{S}(\mathbb{R}^2)$  we define  $K_\xi$  by

$$K_\xi(t) = \psi(t) \int_{\mathbb{R}} \frac{e^{it\tau} - e^{-|t||\xi|^{2\alpha}}}{i\tau + |\xi|^{2\alpha}} \hat{\omega}(\tau) d\tau.$$

Then for all  $\xi \in \mathbb{R}$ ,

$$\left\| \langle i\tau + |\xi|^{2\alpha} \rangle^{\frac{1}{2}} \mathcal{F}_t(K_\xi) \right\|_{L^2(\mathbb{R})}^2 \lesssim \left[ \left( \int_{\mathbb{R}} \frac{|\hat{\omega}(\tau)|}{\langle i\tau + |\xi|^{2\alpha} \rangle} d\tau \right)^2 + \int_{\mathbb{R}} \frac{|\hat{\omega}(\tau)|^2}{\langle i\tau + |\xi|^{2\alpha} \rangle} d\tau \right]. \quad (2.4)$$

**Proposition 2.3** For  $s \in \mathbb{R}$  we have

[a]. for all  $v \in \mathcal{S}(\mathbb{R}^2)$ ,

$$\begin{aligned} & \left\| \chi_{R^+}(t) \psi(t) \int_0^t W(t-t') v(t') dt' \right\|_{X_\alpha^{\frac{1}{2},s}} \\ & \lesssim \|v\|_{X_\alpha^{-\frac{1}{2},s}} + \left( \int_{\mathbb{R}} \langle \xi \rangle^{2s} \left( \int_{\mathbb{R}} \frac{|\hat{v}(\tau)|}{\langle i\tau + |\xi|^{2\alpha} \rangle} d\tau \right)^2 d\xi \right)^{\frac{1}{2}}; \end{aligned} \quad (2.5)$$

[b]. for  $0 < \delta < \frac{1}{2}$  and for all  $v \in X_\alpha^{-\frac{1}{2}+\delta,s}$ ,

$$\left\| \chi_{R^+}(t) \psi(t) \int_0^t W(t-t') v(t') dt' \right\|_{X_\alpha^{\frac{1}{2},s}} \lesssim \|v\|_{X_\alpha^{-\frac{1}{2}+\delta,s}}. \quad (2.6)$$

*Proof.* Assume that  $v \in \mathcal{S}(\mathbb{R}^2)$ . Taking that for  $x$ -Fourier transform we get

$$\begin{aligned} & \chi_{R^+}(t) \psi(t) \int_0^t W(t-t') v(t') dt' \\ &= U(t) \chi_{R^+}(t) \psi(t) \int_{\mathbb{R}} e^{ix\xi} \int_0^t e^{-(t-t')|\xi|^{2\alpha}} \mathcal{F}_x(U(-t') v(t')) dt' d\xi \\ &= U(t) \chi_{R^+}(t) \psi(t) \int_{\mathbb{R}^2} e^{ix\xi} \hat{\omega}(\tau, \xi) e^{-t|\xi|^{2\alpha}} \int_0^t e^{t'|\xi|^{2\alpha}} e^{it'\tau} dt' d\xi d\tau \\ &= U(t) \chi_{R^+}(t) \psi(t) \int_{\mathbb{R}^2} e^{ix\xi} \hat{\omega}(\tau, \xi) \frac{e^{it\tau} - e^{-t|\xi|^{2\alpha}}}{i\tau + |\xi|^{2\alpha}} d\xi d\tau \\ &= U(t) \chi_{R^+}(t) \int_{\mathbb{R}} e^{ix\xi} K_\xi(t) d\xi, \end{aligned}$$

where we denote by  $\omega(t') = U(-t') v(t')$ . By Proposition 2.2, we deduce

$$\begin{aligned} & \left\| \chi_{R^+}(t) \psi(t) \int_0^t W(t-t') v(t') dt' \right\|_{X_\alpha^{\frac{1}{2},s}} \leq \left\| \langle i\tau + |\xi|^{2\alpha} \rangle^{\frac{1}{2}} \langle \xi \rangle^s \mathcal{F}_t(K_\xi(t)) \right\|_{L^2(\mathbb{R}^2)} \\ & \lesssim \left( \int_{\mathbb{R}} \langle \xi \rangle^{2s} \left( \int_{\mathbb{R}} \frac{|\hat{\omega}(\tau)|}{\langle i\tau + |\xi|^{2\alpha} \rangle} d\tau \right)^2 d\xi \right)^{\frac{1}{2}} + \left( \int_{\mathbb{R}} \langle \xi \rangle^{2s} \int_{\mathbb{R}} \frac{|\hat{\omega}(\tau)|^2}{\langle i\tau + |\xi|^{2\alpha} \rangle} d\tau d\xi \right)^{\frac{1}{2}} \\ & \lesssim \left( \int_{\mathbb{R}} \langle \xi \rangle^{2s} \left( \int_{\mathbb{R}} \frac{|\hat{v}(\tau)|}{\langle i\tau + |\xi|^{2\alpha} \rangle} \|e^{-it\xi^3}\|_{L^\infty} d\tau \right)^2 d\xi \right)^{\frac{1}{2}} + \|v\|_{X_\alpha^{-\frac{1}{2},s}} \\ & \lesssim \|v\|_{X_\alpha^{-\frac{1}{2},s}} + \left( \int_{\mathbb{R}} \langle \xi \rangle^{2s} \left( \int_{\mathbb{R}} \frac{|\hat{v}(\tau)|}{\langle i\tau + |\xi|^{2\alpha} \rangle} d\tau \right)^2 d\xi \right)^{\frac{1}{2}}. \end{aligned}$$

We complete the proof of (2.5). Now we prove (2.6). For  $\delta \in (0, \frac{1}{2})$ , obviously

$$\|v\|_{X_\alpha^{-\frac{1}{2}, s}} \leq \|v\|_{X_\alpha^{-\frac{1}{2}+\delta, s}}.$$

By Hölder inequality, we have

$$\int_{\mathbb{R}} \frac{|\hat{v}(\tau)|}{\langle i\tau + |\xi|^{2\alpha} \rangle} d\tau \lesssim \left\| |\hat{v}(\tau)| \langle i\tau + |\xi|^{2\alpha} \rangle^{-\frac{1}{2}+\delta} \right\|_{L^2(\mathbb{R})},$$

and then

$$\left( \int_{\mathbb{R}} \langle \xi \rangle^{2s} \left( \int_{\mathbb{R}} \frac{|\hat{v}(\tau)|}{\langle i\tau + |\xi|^{2\alpha} \rangle} d\tau \right)^2 d\xi \right)^{\frac{1}{2}} \lesssim \|v\|_{X_\alpha^{-\frac{1}{2}+\delta, s}}.$$

□

**Proposition 2.4** *Let  $s \in \mathbb{R}$ ,  $\delta > 0$ . For all  $f \in X_\alpha^{-\frac{1}{2}+\delta, s}$ , one has*

$$t \mapsto \int_0^t W(t-t')f(t')dt' \in C(\mathbb{R}^+, H^{s+2\delta}). \quad (2.7)$$

Moreover, if  $\{f_n\}$  is a sequence with  $f_n \rightarrow 0$  in  $X_\alpha^{-\frac{1}{2}+\delta, s}$  as  $n \rightarrow \infty$ , then

$$\left\| \int_0^t W(t-t')f_n(t')dt' \right\|_{L^\infty(\mathbb{R}^+, H^{s+2\delta})} \rightarrow 0, \quad n \rightarrow \infty. \quad (2.8)$$

*Proof.* The proof is similar to that of Proposition 4 in [7], we omit it. □

### 3 A bilinear estimate and the proof of Theorem 1.1

Let  $Z$  be any abelian additive group with an invariant measure  $d\eta$ . For any integer  $k \geq 2$ , we denote by  $\Gamma_k(Z)$  the hyperplane

$$\Gamma_k(Z) = \{(\eta_1, \dots, \eta_k) \in Z^k : \eta_1 + \dots + \eta_k = 0\},$$

we endow with the obvious measure

$$\int_{\Gamma_k(Z)} f := \int_{Z^{k-1}} f(\eta_1, \dots, \eta_{k-1}, -\eta_1 - \dots - \eta_{k-1}) d\eta_1 \dots d\eta_{k-1}.$$

We define a  $[k; Z]$ -multiplier to be any function  $m : \Gamma_k(Z) \rightarrow \mathbb{C}$ . If  $m$  is a  $[k; Z]$ -multiplier, we define  $\|m\|_{[k; Z]}$  to be the best constant such that the inequality

$$\left| \int_{\Gamma_k(Z)} m(\eta) \Pi_{j=1}^k f_j(\eta_j) \right| \leq \|m\|_{[k; Z]} \Pi_{j=1}^k \|f_j\|_{L^2(Z)},$$

holds for all test functions  $f_j$  on  $Z$ .

In the sequel, we choose  $Z = \mathbb{R} \times \mathbb{R}$ ,  $k = 3$  and  $\eta = (\tau, \xi)$ . For  $N_1, N_2, N_3 > 0$ , we define the quantities  $N_{max} \geq N_{med} \geq N_{min}$  to be the maximum, median and minimum of  $N_1, N_2, N_3$  respectively. Similarly define  $L_{max} \geq L_{med} \geq L_{min}$  whenever  $L_1, L_2, L_3 \geq 1$ . Define

$$h_j(\xi_j) = i\xi_j^3 - |\xi_j|^{2\alpha}, \lambda_j = i\tau_j - h_j(\xi_j), j = 1, 2, 3,$$

and

$$h(\xi) = h_1(\xi_1) + h_2(\xi_2) + h_3(\xi_3).$$

We shall take homogenous dyadic decomposition of the variable  $|\xi_j| \sim N_j > 0$ , and take non-homogenous dyadic decomposition of the variable  $|\lambda_j| \sim L_j \geq 1$  as well as the function  $|h(\xi)| \sim H \geq 1$  ( here the notations  $|\lambda_j| \sim 1$  and  $|h(\xi)| \sim 1$  mean  $|\lambda_j| \leq 1$ ,  $|h(\xi)| \leq 1$ , respectively ). Define

$$X_{N_1, N_2, N_3; H; L_1, L_2, L_3} := \chi_{|h(\xi)| \sim H} \prod_{j=1}^3 \chi_{|\xi_j| \sim N_j} \chi_{|\lambda_j| \sim L_j}.$$

**Lemma 3.1** *Let  $N_1, N_2, N_3 > 0$ ,  $L_1, L_2, L_3 \gtrsim 1$  and  $H \gtrsim 1$  satisfy*

$$N_{max} \sim N_{med}, L_{max} \sim \max\{H, L_{med}\}, H \sim \max\{N_{max}^2 N_{min}, N_{max}^{2\alpha}\}. \quad (3.1)$$

(1). *In the high modulation case  $L_{max} \sim L_{med} \gg H$  we have*

$$\|X_{N_1, N_2, N_3; H; L_1, L_2, L_3}\|_{[3, \mathbb{R} \times \mathbb{R}]} \lesssim L_{min}^{\frac{1}{2}} N_{min}^{\frac{1}{2}}. \quad (3.2)$$

(2). *In the low modulation case  $L_{max} \sim H$ ,*

(2a). *if  $N_{max} \sim N_{med} \sim N_{min}$ , we have*

$$\|X_{N_1, N_2, N_3; H; L_1, L_2, L_3}\|_{[3, \mathbb{R} \times \mathbb{R}]} \lesssim L_{min}^{\frac{1}{2}} \min\{N_{max}^{-\frac{1}{4}} L_{med}^{\frac{1}{4}}, L_{med}^{\frac{1}{4\alpha}}\}; \quad (3.3)$$

(2b). *if  $N_2 \sim N_3 \gg N_1$  and  $H \sim L_1 \geq L_2, L_3$ , we have, for any  $\beta \in (0, 2]$ ,*

$$\|X_{N_1, N_2, N_3; H; L_1, L_2, L_3}\|_{[3, \mathbb{R} \times \mathbb{R}]} \lesssim L_{min}^{\frac{1}{2}} \min\{N_1^{\frac{1}{2}}, L_{med}^{\frac{1}{4\alpha}}, N_2^{\frac{\beta-2}{2\beta}} N_1^{-\frac{1}{2\beta}} L_{med}^{\frac{1}{2\beta}}\}; \quad (3.4)$$

*Similarly for permutations;*

(2c). *In all other cases, we have*

$$\|X_{N_1, N_2, N_3; H; L_1, L_2, L_3}\|_{[3, \mathbb{R} \times \mathbb{R}]} \lesssim L_{min}^{\frac{1}{2}} \min\{N_{max}^{-1} L_{med}^{\frac{1}{2}}, L_{med}^{\frac{1}{4\alpha}}, N_{min}^{\frac{1}{2}}\}. \quad (3.5)$$

*Proof.* We consider the high modulation case  $L_{max} \sim L_{med} \gg H$ . By using the comparison principle (Lemma 3.1 in [10]), we have (without loss of generality we assume  $L_1 \geq L_2 \geq L_3$  and  $N_1 \geq N_2 \geq N_3$ )

$$\|X_{N_1, N_2, N_3; H; L_1, L_2, L_3}\|_{[3, \mathbb{R} \times \mathbb{R}]} \lesssim \|\chi_{|\lambda_3| \sim L_3} \chi_{|\xi_3| \sim N_3}\|_{[3, \mathbb{R} \times \mathbb{R}]} \quad (3.6)$$

By Lemma 3.14 and Lemma 3.6 in [10],

$$\|\chi_{|\lambda_3| \sim L_3} \chi_{|\xi_3| \sim N_3}\|_{[3, \mathbb{R} \times \mathbb{R}]} \lesssim \|\chi_{|\lambda_3| \sim L_3}\|_{[3, \mathbb{R}]} \|\chi_{|\xi_3| \sim N_3}\|_{[3, \mathbb{R}]} \lesssim L_3^{\frac{1}{2}} N_3^{\frac{1}{2}}. \quad (3.7)$$

Although we derived (3.7) assuming  $L_1 \geq L_2 \geq L_3$  and  $N_1 \geq N_2 \geq N_3$ , it is clear from symmetry that

$$\|X_{N_1, N_2, N_3; H; L_1, L_2, L_3}\|_{[3, \mathbb{R} \times \mathbb{R}]} \lesssim L_{min}^{\frac{1}{2}} N_{min}^{\frac{1}{2}}. \quad (3.8)$$

We now consider the low modulation case  $H \sim L_{max}$ . Suppose for the moment that  $N_1 \geq N_2 \geq N_3$ . The  $\xi_3$  variable is currently localized to the annulus  $\{|\xi_3| \sim N_3\}$ . By a finite partition of unity we can restrict it further to a ball  $\{|\xi_3 - \xi_3^0| \ll N_3\}$  for some  $|\xi_3^0| \sim N_3$ . Then by Box Localization ( Lemma 3.13 in [10] ) we may localize  $\xi_1, \xi_2$  similarly to regions  $\{|\xi_1 - \xi_1^0| \ll N_3\}$  and  $\{|\xi_2 - \xi_2^0| \ll N_3\}$  where  $|\xi_j^0| \sim N_j$ . We may assume that  $|\xi_1^0 + \xi_2^0 + \xi_3^0| \ll N_3$  since we have  $\xi_1 + \xi_2 + \xi_3 = 0$ . We summarize this symmetrically as

$$\|X_{N_1, N_2, N_3; H; L_1, L_2, L_3}\|_{[3, \mathbb{R} \times \mathbb{R}]} \lesssim \left\| \chi_{|h(\xi)| \sim H} \prod_{j=1}^3 \chi_{|\lambda_j| \sim L_j} \chi_{|\xi_j - \xi_j^0| \ll N_{min}} \right\|_{[3, \mathbb{R} \times \mathbb{R}]}, \quad (3.9)$$

for some  $\xi_1^0, \xi_2^0, \xi_3^0$  satisfying

$$|\xi_j^0| \sim N_j, |\xi_1^0 + \xi_2^0 + \xi_3^0| \ll N_{min}.$$

Without loss of generality, we assume  $L_1 \geq L_2 \geq L_3$ . By Lemma 3.6, Lemma 3.1 and Corollary 3.10 in [10] we get

$$\begin{aligned} & \|X_{N_1, N_2, N_3; H; L_1, L_2, L_3}\|_{[3, \mathbb{R} \times \mathbb{R}]} \\ & \lesssim \left\| \chi|h(\xi)| \sim_H \Pi_{j=2}^3 \chi_{|\xi_j - \xi_j^0| \ll N_{min}} \chi_{|\lambda_j| \sim L_j} \right\|_{[3, \mathbb{R} \times \mathbb{R}]} \\ & \lesssim \left| \{(\tau_2, \xi_2) : |\xi_2 - \xi_2^0| \ll N_{min}, |i\tau_2 - h_2(\xi_2)| \sim L_2, \right. \\ & \quad \left. |\xi - \xi_2 - \xi_3^0| \ll N_{min}, |i(\tau - \tau_2) - h_3(\xi - \xi_2)| \sim L_3\} \right|^{\frac{1}{2}} \end{aligned} \quad (3.10)$$

for some  $(\tau, \xi) \in \mathbb{R} \times \mathbb{R}$ . For fixed  $\xi_2$ , the set of possible  $\tau_2$  ranges in an interval of length  $O(\min\{L_2, L_3\})$ , and vanishes unless

$$|i\tau - h_2(\xi_2) - h_3(\xi - \xi_2)| = O(\max\{L_2, L_3\}).$$

Then we get, for some  $(\tau, \xi) \in \mathbb{R} \times \mathbb{R}$ ,

$$\begin{aligned} & \|X_{N_1, N_2, N_3; H; L_1, L_2, L_3}\|_{[3, \mathbb{R} \times \mathbb{R}]} \lesssim L_3^{\frac{1}{2}} |\{\xi_2 : |\xi_2 - \xi_2^0| \ll N_{min}, \\ & \quad |\xi - \xi_2 - \xi_3^0| \ll N_{min}, |i\tau - h_2(\xi_2) - h_3(\xi - \xi_2)| = O(L_2)\}|^{\frac{1}{2}}. \end{aligned}$$

Note that the inequality  $|\xi - \xi_2 - \xi_3^0| \ll N_{min}$  implies  $|\xi - \xi_1^0| \ll N_{min}$ . Then we have

$$\begin{aligned} & \|X_{N_1, N_2, N_3; H; L_1, L_2, L_3}\|_{[3, \mathbb{R} \times \mathbb{R}]} \\ & \lesssim L_3^{\frac{1}{2}} |\{\xi_2 : |\xi_2 - \xi_2^0| \ll N_{min}, |\xi - \xi_1^0| \ll N_{min}, \\ & \quad |i\tau - h_2(\xi_2) - h_3(\xi - \xi_2)| = O(L_2)\}|^{\frac{1}{2}}. \end{aligned} \quad (3.11)$$

To compute the right-hand side of the expression (3.11) we use the identity

$$|i\tau - h_2(\xi_2) - h_3(\xi - \xi_2)| = \left| i\tau - 3i\xi(\xi_2 - \frac{\xi}{2})^2 + i\frac{\xi^3}{4} + (|\xi_2|^{2\alpha} + |\xi_2 - \xi|^{2\alpha}) \right| = O(L_2),$$

which implies

$$3\xi(\xi_2 - \frac{\xi}{2})^2 + \frac{\xi^3}{4} = \tau + O(L_2) \quad (3.12)$$

and

$$|\xi_2|^{2\alpha} + |\xi_2 - \xi|^{2\alpha} = O(L_2). \quad (3.13)$$

We need only consider three cases:  $N_1 \sim N_2 \sim N_3$ ,  $N_1 \sim N_2 \gg N_3$ , and  $N_2 \sim N_3 \gg N_1$ . (The case  $N_1 \sim N_3 \gg N_2$  then follows by symmetry).

If  $N_1 \sim N_2 \sim N_3$ , by  $|\xi - \xi_1^0| \ll N_{min}$  we deduce  $|\xi| \sim N_1$ . we see from (3.12) that  $\xi_2$  variable is contained in the union of two intervals of length  $O(N_1^{\frac{1}{2}} L_2^{\frac{1}{2}})$  at worst, and from (3.13) that  $|\xi_2| \leq L_2^{\frac{1}{2\alpha}}$ , and (3.3) follows from (3.11).

If  $N_1 \sim N_2 \gg N_3$ , by  $|\xi - \xi_1^0| \ll N_{min}$ ,  $|\xi_2 - \xi_2^0 - \frac{\xi - \xi_1^0}{2} - \xi_3^0| \ll N_{min}$  and

$$\left| \xi_2 - \frac{\xi}{2} \right| = \left| \xi_2 - \xi_2^0 - \frac{\xi - \xi_1^0}{2} - \xi_3^0 - \frac{\xi_1^0}{2} \right|$$

we get  $|\xi| \sim N_1$  and  $|\xi_2 - \frac{\xi}{2}| \sim N_1$ . we see from (3.12) that  $\xi_2$  variable is contained in the union of two intervals of length  $O(N_1^{-2} L_2)$  at worst, and from (3.13) that  $|\xi_2| \leq L_2^{\frac{1}{2\alpha}}$ , and (3.5) follows from (3.11).

If  $N_2 \sim N_3 \gg N_1$ , then we must have  $|\xi| \sim N_1$  and  $|\xi_2 - \frac{\xi}{2}| \sim N_2$ . For a given  $\beta \in (0, 2]$ , we have  $|\xi||\xi_2 - \frac{\xi}{2}|^{2-\beta} \sim N_1 N_2^{2-\beta}$ . we see from (3.12) that  $\xi_2$  variable is contained in the union of two intervals of length  $O(N_1^{-\frac{1}{\beta}} N_2^{\frac{\beta-2}{\beta}} L_2^{\frac{1}{\beta}})$  at worst, and from (3.13) that  $|\xi_2| \leq L_2^{\frac{1}{2\alpha}}$ . (3.4) follows from (3.11) and the fact that  $|\xi_2 - \xi_2^0| \ll N_1$  for some  $|\xi_2^0| \ll N_2$ .  $\square$

**Lemma 3.2** *For a given  $\rho \in (\frac{1}{2}, \min\{\frac{3+2\alpha}{4}, 1\})$  and for any  $\delta > 0$  small we have*

$$\left\| \frac{\xi_3 < \xi_1 >^\rho < \xi_2 >^\rho < \xi_3 >^{-\rho}}{< \lambda_1 >^{\frac{1}{2}} < \lambda_2 >^{\frac{1}{2}} < \lambda_3 >^{\frac{1}{2}-\delta}} \right\|_{[3, \mathbb{R} \times \mathbb{R}]} \lesssim 1. \quad (3.14)$$

*Proof.* We have

$$\begin{aligned} & \left\| \frac{\xi_3 < \xi_1 >^\rho < \xi_2 >^\rho < \xi_3 >^{-\rho} \chi_{|\xi_1| \lesssim 1} \chi_{|\xi_2| \lesssim 1} \chi_{|\xi_3| \lesssim 1}}{< \lambda_1 >^{\frac{1}{2}} < \lambda_2 >^{\frac{1}{2}} < \lambda_3 >^{\frac{1}{2}-\delta}} \right\|_{[3, \mathbb{R} \times \mathbb{R}]} \\ & \lesssim \left\| \frac{\chi_{|\xi_1| \lesssim 1} \chi_{|\xi_2| \lesssim 1} \chi_{|\xi_3| \lesssim 1}}{< \lambda_1 >^{\frac{1}{2}} < \lambda_2 >^{\frac{1}{2}} < \lambda_3 >^{\frac{1}{2}-\delta}} \right\|_{[3, \mathbb{R} \times \mathbb{R}]} . \end{aligned}$$

By taking the non-homogenous dyadic decomposition of the variable  $|\lambda_j| \sim L_j \geq 1$  ( here the notation  $|\lambda_j| \sim L_j = 1$  means  $|\lambda_j| \leq 1$ ), we get

$$\begin{aligned} & \left\| \frac{\xi_3 < \xi_1 >^\rho < \xi_2 >^\rho < \xi_3 >^{-\rho} \chi_{|\xi_1| \lesssim 1} \chi_{|\xi_2| \lesssim 1} \chi_{|\xi_3| \lesssim 1}}{< \lambda_1 >^{\frac{1}{2}} < \lambda_2 >^{\frac{1}{2}} < \lambda_3 >^{\frac{1}{2}-\delta}} \right\|_{[3, \mathbb{R} \times \mathbb{R}]} \\ & \lesssim \sum_{L_1, L_2, L_3 \gtrsim 1} \frac{\left\| \prod_{j=1}^3 \chi_{|\xi_j| \lesssim 1} \chi_{|\lambda_j| \sim L_j} \right\|_{[3, \mathbb{R} \times \mathbb{R}]}}{< L_1 >^{\frac{1}{2}} < L_2 >^{\frac{1}{2}} < L_3 >^{\frac{1}{2}-\delta}} \\ & \lesssim \sum_{L_1, L_2, L_3 \gtrsim 1} \frac{L_{\min}^{\frac{1}{2}}}{< L_1 >^{\frac{1}{2}} < L_2 >^{\frac{1}{2}} < L_3 >^{\frac{1}{2}-\delta}} \\ & \lesssim \sum_{L_{\min}, L_{\text{med}}, L_{\max} \gtrsim 1} \frac{1}{< L_{\text{med}} >^{\frac{1}{2}} < L_{\max} >^{\frac{1}{2}-\delta}} \lesssim 1, \end{aligned} \quad (3.15)$$

here we have used the estimate (without loss of generality we assume  $L_1 \lesssim L_2 \lesssim L_3$ )

$$\left\| \prod_{j=1}^3 \chi_{|\xi_j| \lesssim 1} \chi_{|\lambda_j| \sim L_j} \right\|_{[3, \mathbb{R} \times \mathbb{R}]} \lesssim \left\| \chi_{|\xi_1| \lesssim 1} \left\| \chi_{|i\tau_1 - h_1(\xi)| \sim L_1} \right\|_{[3, \mathbb{R}]} \right\|_{[3, \mathbb{R}]} \lesssim L_1^{\frac{1}{2}}.$$

What remains is to estimate the term

$$\left\| \frac{\xi_3 < \xi_1 >^\rho < \xi_2 >^\rho < \xi_3 >^{-\rho} \chi_{\max\{|\xi_1|, |\xi_2|, |\xi_3|\} \gtrsim 1}}{< \lambda_1 >^{\frac{1}{2}} < \lambda_2 >^{\frac{1}{2}} < \lambda_3 >^{\frac{1}{2}-\delta}} \right\|_{[3, \mathbb{R} \times \mathbb{R}]} . \quad (3.16)$$

By taking the homogenous dyadic decomposition of the variable  $|\xi_j| \sim N_j > 0$ , by taking the non-homogenous dyadic decomposition of the variable  $|\lambda_j| \sim L_j \geq 1$ , and the function  $|h(\xi)| \sim H \geq 1$  ( here the notation  $|\lambda_j| \sim L_j = 1$ ,  $|h(\xi)| \sim H = 1$  means  $|\lambda_j| \leq 1$ ,  $|h(\xi)| \leq 1$ , respectively), we have

$$\begin{aligned} & \left\| \frac{\xi_3 < \xi_1 >^\rho < \xi_2 >^\rho < \xi_3 >^{-\rho} \chi_{\max\{|\xi_1|, |\xi_2|, |\xi_3|\} \gtrsim 1}}{< \lambda_1 >^{\frac{1}{2}} < \lambda_2 >^{\frac{1}{2}} < \lambda_3 >^{\frac{1}{2}-\delta}} \right\|_{[3, \mathbb{R} \times \mathbb{R}]} \\ & \lesssim \left\| \sum_{N_{\max} \gtrsim 1} \sum_{L_1, L_2, L_3 \geq 1} \sum_{H \geq 1} \frac{N_3 < N_1 >^\rho < N_2 >^\rho}{< N_3 >^\rho L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} L_3^{\frac{1}{2}-\delta}} X_{N_1, N_2, N_3; H; L_1, L_2, L_3} \right\|_{[3, \mathbb{R} \times \mathbb{R}]} , \end{aligned} \quad (3.17)$$



where  $X_{N_1, N_2, N_3; H; L_1, L_2, L_3}$  is the multiplier

$$X_{N_1, N_2, N_3; H; L_1, L_2, L_3} := \chi_{|h(\xi)| \sim H} \prod_{j=1}^3 \chi_{|\xi_j| \sim N_j} \chi_{|\lambda_j| \sim L_j}.$$

From the identities  $\xi_1 + \xi_2 + \xi_3 = 0$  and  $\tau_1 + \tau_2 + \tau_3 = 0$  we see that

$$h(\xi) = -\lambda_1 - \lambda_2 - \lambda_3 = 3i\xi_1\xi_2\xi_3 - (|\xi_1|^{2\alpha} + |\xi_2|^{2\alpha} + |\xi_3|^{2\alpha}).$$

Then the multiplier  $X_{N_1, N_2, N_3; H; L_1, L_2, L_3}$  vanishes unless

$$N_{max} \sim N_{med}, L_{max} \sim \max\{H, L_{med}\}, H \sim \max\{N_{max}^2 N_{min}, N_{max}^{2\alpha}\}. \quad (3.18)$$

Thus we may implicitly assume (3.18) in the summations. By applying Schur's test (Lemma 3.11 in [10]),

$$(3.17) \lesssim \sup_{N \gtrsim 1} \left\| \sum_{N_{max} \sim N_{med} \sim N} \sum_{H \gtrsim 1} \sum_{L_{max} \sim \max\{H, L_{med}\}} \frac{N_3 < N_1 >^\rho < N_2 >^\rho}{< N_3 >^\rho L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} L_3^{\frac{1}{2}-\delta}} X_{N_1, N_2, N_3; H; L_1, L_2, L_3} \right\|_{[3, \mathbb{R} \times \mathbb{R}]} \quad (3.19)$$

In light of (3.18) and the comparison principle in [10], we thus see that at least one of the inequalities

$$(3.19) \lesssim \sup_{N \gtrsim 1} \sum_{N_{max} \sim N_{med} \sim N} \sum_{L_{max} \gtrsim L_{med} \gtrsim L_{min}} \frac{N_3 < N_1 >^\rho < N_2 >^\rho}{< N_3 >^\rho L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} L_3^{\frac{1}{2}-\delta}} \|X_{N_1, N_2, N_3; L_{max}; L_1, L_2, L_3}\|_{[3, R \times R]}, \quad (3.20)$$

or

$$(3.19) \lesssim \sup_{N \gtrsim 1} \sum_{N_{max} \sim N_{med} \sim N} \sum_{L_{max} \sim L_{med}} \sum_{H \ll L_{max}} \frac{N_3 < N_1 >^\rho < N_2 >^\rho}{< N_3 >^\rho L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} L_3^{\frac{1}{2}-\delta}} \|X_{N_1, N_2, N_3; H; L_1, L_2, L_3}\|_{[3, \mathbb{R} \times \mathbb{R}]} \quad (3.21)$$

holds. It is sufficient to prove (3.20)  $\lesssim 1$  and (3.21)  $\lesssim 1$ .

**The proof of (3.21)  $\lesssim 1$ .** Note that the inequality  $N_{max}^2 N_{min} \geq N_{max}^{2\alpha}$  implies  $N_{min} \geq N_{max}^{2\alpha-2}$ . When  $N_{min} \gtrsim 1$ , by using the estimate (1) in Lemma 3.1, we get from (3.17) and (3.18),

$$\begin{aligned} (3.21) &\lesssim \sup_{N \gtrsim 1} \sum_{N_{max} \sim N_{med} \sim N} \sum_{L_{max} \sim L_{med}} \sum_{H \sim \max\{N_{max}^2 N_{min}, N_{max}^{2\alpha}\} \ll L_{max}} \frac{N_3 < N_1 >^\rho < N_2 >^\rho}{< N_3 >^\rho L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} L_3^{\frac{1}{2}-\delta}} L_{min}^{\frac{1}{2}} N_{min}^{\frac{1}{2}} \\ &\lesssim \sup_{N \gtrsim 1} \sum_{N_{max} \sim N_{med} \sim N, N_{min} \gtrsim 1} \sum_{L_{max} \sim L_{med}} \sum_{H \sim N_{max}^2 N_{min} \ll L_{max}} \frac{N_{min}^{1-\rho} N_{max}^{2\rho}}{L_{min}^{\frac{1}{2}} L_{max}^{1-\delta}} L_{min}^{\frac{1}{2}} N_{min}^{\frac{1}{2}} \\ &\lesssim \sup_{N \gtrsim 1} \sum_{N_{max} \sim N_{med} \sim N, N_{min} \gtrsim 1} \sum_{L_{max} \sim L_{med} \gtrsim N^2 N_{min}} \frac{N_{min}^{\frac{1}{2}-\rho+2\delta} N^{2\rho-2+4\delta}}{L_{max}^\delta} \log_2(L_{max}) \\ &\lesssim \sup_{N \gtrsim 1} \sum_{N_{max} \sim N_{med} \sim N, N_{min} \gtrsim 1} N_{min}^{\frac{1}{2}-\rho+2\delta} N^{2\rho-2+4\delta} \\ &\lesssim \sup_{N \gtrsim 1} N^{2\rho-2+4\delta} \lesssim 1, \end{aligned} \quad (3.22)$$

for  $\frac{1}{2} < \rho < 1$  and  $\delta > 0$  small.

When  $N_1 \sim N_2 \gg N_3$  with  $N_3 \lesssim 1$ , by using the estimate (1) in Lemma 3.1, we get from (3.17) and (3.18),

$$\begin{aligned}
(3.21) &\lesssim \sup_{N \gtrsim 1} \sum_{N_1 \sim N_2 \sim N, N_3 \lesssim 1} \sum_{L_{max} \sim L_{med}} \sum_{H \sim \max\{N^2 N_3, N^{2\alpha}\} \ll L_{max}} \frac{N_3 < N_1 >^\rho < N_2 >^\rho}{< N_3 >^\rho L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} L_3^{\frac{1}{2}-\delta}} L_{min}^{\frac{1}{2}} N_3^{\frac{1}{2}} \\
&\lesssim \sup_{N \gtrsim 1} \sum_{N_1 \sim N_2 \sim N, 1 \gtrsim N_3 \geq N^{2\alpha-2}} \sum_{L_{max} \sim L_{med}} \sum_{H \sim N^2 N_3 \ll L_{max}} \frac{N_3 N^{2\rho}}{L_{min}^{\frac{1}{2}} L_{max}^{1-\delta}} L_{min}^{\frac{1}{2}} N_3^{\frac{1}{2}} \\
&\quad + \sup_{N \gtrsim 1} \sum_{N_1 \sim N_2 \sim N, N_3 \leq N^{2\alpha-2}} \sum_{L_{max} \sim L_{med}} \sum_{H \sim N^{2\alpha} \ll L_{max}} \frac{N_3 N^{2\rho}}{L_{min}^{\frac{1}{2}} L_{max}^{1-\delta}} L_{min}^{\frac{1}{2}} N_3^{\frac{1}{2}} \\
&\lesssim \sup_{N \gtrsim 1} \sum_{N_1 \sim N_2 \sim N, 1 \gtrsim N_3 \geq N^{2\alpha-2}} \sum_{L_{max} \sim L_{med} \gtrsim N^2 N_3} \frac{N_3^{\frac{1}{2}+2\delta} N^{2\rho-2+4\delta}}{L_{max}^\delta} \log_2(L_{max}) \\
&\quad + \sup_{N \gtrsim 1} \sum_{N_1 \sim N_2 \sim N, N_3 \leq N^{2\alpha-2}} \sum_{L_{max} \sim L_{med} \gtrsim N^{2\alpha}} \frac{N_3^{\frac{3}{2}} N^{2\rho-2\alpha+2\alpha\delta}}{L_{max}^\delta} \log_2(L_{max}) \\
&\lesssim \sup_{N \gtrsim 1} \sum_{N_1 \sim N_2 \sim N, 1 \gtrsim N_3 \geq N^{2\alpha-2}} N_3^{\frac{1}{2}+2\delta} N^{2\rho-2+4\delta} \\
&\quad + \sup_{N \gtrsim 1} \sum_{N_1 \sim N_2 \sim N, N_3 \leq N^{2\alpha-2}} N_3^{\frac{3}{2}} N^{2\rho-2\alpha+2\alpha\delta} \\
&\lesssim \sup_{N \gtrsim 1} N^{2\rho-2+4\delta} + \sup_{N \gtrsim 1} N^{2\rho+\alpha-3+2\delta} \lesssim 1, \tag{3.23}
\end{aligned}$$

for  $\delta > 0$  small, since  $\frac{1}{2} < \rho < 1$  and  $0 \leq \alpha \leq 1$  imply

$$2\rho + \alpha - 3 + 2\delta < 0, \quad 2\rho - 2 + 4\delta < 0$$

for  $\delta > 0$  small.

When  $\alpha = 1$  and  $N_1 \sim N_3 \sim N \gg N_2$  with  $N_2 \lesssim 1$ , we have  $N^{2\alpha} \geq N^2 N_{min}$ . By using the estimate (1) in Lemma 3.1, we get from (3.17) and (3.18),

$$\begin{aligned}
(3.21) &\lesssim \sup_{N \gtrsim 1} \sum_{N_1 \sim N_3 \sim N, N_2 \lesssim 1} \sum_{L_{max} \sim L_{med}} \sum_{H \sim N^{2\alpha} \ll L_{max}} \frac{N}{L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} L_3^{\frac{1}{2}-\delta}} L_{min}^{\frac{1}{2}} N_2^{\frac{1}{2}} \\
&\lesssim \sup_{N \gtrsim 1} \sum_{N_1 \sim N_3 \sim N, N_2 \lesssim 1} \sum_{L_{max} \sim L_{med} \gg N^2} \frac{N}{L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} L_3^{\frac{1}{2}-\delta}} L_{min}^{\frac{1}{2}} N_2^{\frac{1}{2}} \\
&\lesssim \sup_{N \gtrsim 1} \sum_{N_2 \lesssim 1} \sum_{L_{max} \sim L_{med} \gg N^2} \frac{N N_2^{\frac{1}{2}}}{L_{med}^{\frac{1}{2}} L_{max}^{\frac{1}{2}-\delta}} \\
&\lesssim \sup_{N \gtrsim 1} \sum_{N_2 \lesssim 1} \sum_{L_{max} \sim L_{med} \gg N^2} \frac{N^{4\delta-1} N_2^{\frac{1}{2}}}{L_{max}^\delta} \\
&\lesssim \sup_{N \gtrsim 1} \sum_{N_2 \lesssim 1} N^{4\delta-1} N_2^{\frac{1}{2}} \lesssim 1 \tag{3.24}
\end{aligned}$$

for  $\delta > 0$  small.

When  $0 \leq \alpha < 1$  and  $N_1 \sim N_3 \sim N \gg N_2$  with  $N_2 \lesssim 1$ , by using the estimate (1) in Lemma

3.1, we get from (3.17) and (3.18),

$$\begin{aligned}
(3.21) &\lesssim \sup_{N \gtrsim 1} \sum_{N_1 \sim N_3 \sim N} \sum_{L_{max} \sim L_{med}} \sum_{H \sim \max\{N^2 N_2, N^{2\alpha}\} \ll L_{max}} \frac{N}{L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} L_3^{\frac{1}{2}-\delta}} L_{min}^{\frac{1}{2}} N_2^{\frac{1}{2}} \\
&\lesssim \sup_{N \gtrsim 1} \sum_{N_1 \sim N_3 \sim N, N_2 \geq N^{2\alpha-2}} \sum_{L_{max} \sim L_{med} \gg N^2 N_2} \frac{N}{L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} L_3^{\frac{1}{2}-\delta}} L_{min}^{\frac{1}{2}} N_2^{\frac{1}{2}} \\
&\quad + \sup_{N \gtrsim 1} \sum_{N_1 \sim N_3 \sim N, N_2 \leq N^{2\alpha-2}} \sum_{L_{max} \sim L_{med} \gg N^{2\alpha}} \frac{N}{L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} L_3^{\frac{1}{2}-\delta}} L_{min}^{\frac{1}{2}} N_2^{\frac{1}{2}} \\
&\lesssim \sup_{N \gtrsim 1} \sum_{N_2 \geq N^{2\alpha-2}} \sum_{L_{max} \sim L_{med} \gg N^2 N_2} \frac{N N_2^{\frac{1}{2}}}{L_{med}^{\frac{1}{2}} L_{max}^{\frac{1}{2}-\delta}} \\
&\quad + \sup_{N \gtrsim 1} \sum_{N_2 \leq N^{2\alpha-2}} \sum_{L_{max} \sim L_{med} \gg N^{2\alpha}} \frac{N N_2^{\frac{1}{2}}}{L_{med}^{\frac{1}{2}} L_{max}^{\frac{1}{2}-\delta}} \\
&\lesssim \sup_{N \gtrsim 1} \sum_{N_2 \geq N^{2\alpha-2}} \sum_{L_{max} \sim L_{med} \gg N^2 N_2} \frac{N^{-1+4\delta} N_2^{-\frac{1}{2}+2\delta}}{L_{max}^{\delta}} \\
&\quad + \sup_{N \gtrsim 1} \sum_{N_2 \leq N^{2\alpha-2}} \sum_{L_{max} \sim L_{med} \gg N^{2\alpha}} \frac{N^{4\alpha\delta-2\alpha+1} N_2^{\frac{1}{2}}}{L_{max}^{\delta}} \\
&\lesssim \sup_{N \gtrsim 1} \sum_{N_2 \geq N^{2\alpha-2}} N^{-1+4\delta} N_2^{-\frac{1}{2}+2\delta} + \sup_{N \gtrsim 1} \sum_{N_2 \leq N^{2\alpha-2}} N^{4\alpha\delta-2\alpha+1} N_2^{\frac{1}{2}} \\
&\lesssim \sup_{N \gtrsim 1} N^{-\alpha+4\alpha\delta} \lesssim 1
\end{aligned} \tag{3.25}$$

for  $\delta > 0$  small. By symmetric we know the estimate (3.21)  $\lesssim 1$  holds when  $N_2 \sim N_3 \sim N \gg N_1$  and  $N_1 \lesssim 1$ .

**The proof of (3.20)  $\lesssim 1$ .** We first deal with the contribution where the case (2a) in Lemma 3.1 holds. In this case we have  $N_1 \sim N_2 \sim N_3 \sim N$ ,  $L_{max} \sim N^3$  and  $L_{min} \gtrsim N^{2\alpha}$ , since we have  $L_j \sim |\lambda_j| \geq |\xi_j|^{2\alpha}$ . So we get

$$\begin{aligned}
(3.20) &\lesssim \sup_{N \gtrsim 1} \sum_{L_{med} \gtrsim N^{2\alpha}, L_{max} \sim N^3} \frac{N^{1+\rho}}{L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} L_3^{\frac{1}{2}-\delta}} L_{min}^{\frac{1}{2}} \min\{N^{-\frac{1}{4}} L_{med}^{\frac{1}{4}}, L_{med}^{\frac{1}{4\alpha}}\} \\
&\lesssim \sup_{N \gtrsim 1} \sum_{L_{med} \gtrsim N^{2\alpha}, L_{max} \sim N^3} \frac{N^{\frac{3}{4}+\rho}}{L_{med}^{\frac{1}{4}} L_{max}^{\frac{1}{2}-\delta}} \lesssim \sup_{N \gtrsim 1} \sum_{L_{max} \sim N^3} \frac{N^{-\frac{3}{4}+\rho-\frac{\alpha}{2}+6\delta}}{L_{max}^{\delta}} \\
&\lesssim \sup_{N \gtrsim 1} N^{-\frac{3}{4}+\rho-\frac{\alpha}{2}+6\delta} \lesssim 1
\end{aligned} \tag{3.26}$$

for  $\delta > 0$  small, since we have  $\rho < \frac{3+2\alpha}{4}$ .

Second, we deal with the contribution where the case (2b) in Lemma 3.1 applies. We choose  $\beta > 0$  small in (3.4). We do not have perfect symmetry and must consider the cases

**Case A:**  $N \sim N_1 \sim N_2 \gg N_3 \gtrsim 1; H \sim L_3 \gtrsim L_1, L_2,$

**Case B:**  $N \sim N_1 \sim N_2 \gg N_3, N_3 \lesssim 1; H \sim L_3 \gtrsim L_1, L_2,$

**Case C:**  $N \sim N_1 \sim N_3 \gg N_2 \gtrsim 1; H \sim L_2 \gtrsim L_1, L_3,$

**Case D:**  $N \sim N_1 \sim N_3 \gg N_2, N_2 \lesssim 1; H \sim L_2 \gtrsim L_1, L_3,$

separately.

**The estimate in Case A.** In this case, we have  $L_{max} \sim N^2 N_3$  and  $L_{med} \gtrsim N^{2\alpha}$ , and then  $N_3^{\frac{1}{2}} \ll N^{\frac{1}{2}} \leq L_{med}^{\frac{1}{2\alpha}}$ . When  $0 \leq \alpha < 1$ , we have  $N_3^{\beta+1} N^{2-\beta} \geq N^{2\alpha}$  for  $N_3 \gtrsim 1$  and  $\beta > 0$  small.

When  $L_{med} \geq N_3^{\beta+1} N^{2-\beta} \geq N^{2\alpha}$ , we get from (3.4) and (3.20) that

$$\begin{aligned}
(3.20) &\lesssim \sup_{N \gtrsim 1} \sum_{N \gg N_3 \gtrsim 1} \sum_{L_{max} \sim N^2 N_3, L_{med} \geq N_3^{\beta+1} N^{2-\beta}} \frac{N_3^{1-\rho} N^{2\rho}}{L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} L_3^{\frac{1}{2}-\delta}} L_{min}^{\frac{1}{2}} N_3^{\frac{1}{2}} \\
&\lesssim \sup_{N \gtrsim 1} \sum_{N \gg N_3 \gtrsim 1} \sum_{L_{max} \sim N^2 N_3, L_{med} \geq N_3^{\beta+1} N^{2-\beta}} \frac{N_3^{1-\rho+\delta} N^{2\rho-1+2\delta}}{L_{med}^{\frac{1}{2}}} \\
&\lesssim \sup_{N \gtrsim 1} \sum_{N \gg N_3 \gtrsim 1} \sum_{L_{med} \lesssim L_{max} \sim N^2 N_3} \frac{N_3^{\frac{1}{2}-\frac{\beta}{2}-\rho+\delta(2+\beta)} N^{2\rho-2+\frac{\beta}{2}+\delta(4-\beta)}}{L_{med}^{\delta}} \\
&\lesssim \sup_{N \gtrsim 1} \sum_{N \gg N_3 \gtrsim 1} N_3^{\frac{1}{2}-\frac{\beta}{2}-\rho+\delta(2+\beta)} N^{2\rho-2+\frac{\beta}{2}+\delta(4-\beta)} \\
&\lesssim \sup_{N \gtrsim 1} N^{2\rho-2+\frac{\beta}{2}+\delta(4-\beta)} \lesssim 1,
\end{aligned} \tag{3.27}$$

for  $\delta > 0$  and  $\beta > 0$  small, since the inequality  $\frac{1}{2} < \rho < 1$  implies

$$2\rho - 2 + \frac{\beta}{2} + \delta(4 - \beta) < 0, \quad \frac{1}{2} - \frac{\beta}{2} - \rho + \delta(2 + \beta) < 0$$

for  $\delta > 0$  and  $\beta > 0$  small.

When  $L_{med} \leq N_3^{\beta+1} N^{2-\beta}$  and  $L_{med} \gtrsim N^{2\alpha}$ , We get from (3.4) and (3.20) that

$$\begin{aligned}
(3.20) &\lesssim \sup_{N \gtrsim 1} \sum_{N_3 \gtrsim 1} \sum_{L_{med} \leq N_3^{\beta+1} N^{2-\beta}, L_3 \sim N^2 N_3} \frac{N_3^{1-\rho} N^{2\rho}}{L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} L_3^{\frac{1}{2}-\delta}} L_{min}^{\frac{1}{2}} N_3^{-\frac{1}{2\beta}} N^{\frac{\beta-2}{2\beta}} L_{med}^{\frac{1}{2\beta}} \\
&\lesssim \sup_{N \gtrsim 1} \sum_{N_3 \gtrsim 1} \sum_{L_{max} \sim N^2 N_3} \frac{N_3^{\frac{1}{2}-\rho+(2+\beta)\delta-\frac{\beta}{2}} N^{2\rho-2+2\delta+\frac{\beta}{2}+(2-\beta)\delta}}{L_{med}^{\delta}} \\
&\lesssim \sup_{N \gtrsim 1} \sum_{N_3 \gtrsim 1} N_3^{\frac{1}{2}-\rho+(2+\beta)\delta-\frac{\beta}{2}} N^{2\rho-2+2\delta+\frac{\beta}{2}+(2-\beta)\delta} \lesssim 1
\end{aligned} \tag{3.28}$$

for  $\delta > 0$  and  $\beta > 0$  small, since the inequality  $\frac{1}{2} < \rho < 1$  means

$$\frac{1}{2} - \rho + (2 + \beta)\delta - \frac{\beta}{2} < 0, \quad 2\rho - 2 + 2\delta + \frac{\beta}{2} + (2 - \beta)\delta < 0$$

for  $\delta > 0$  and  $\beta > 0$  small.

When  $\alpha = 1$ , we must consider the case  $L_{med} \geq N_3^{\beta+1} N^{2-\beta}$ ,  $L_{med} \gtrsim N^2$  and  $N_3^{\beta+1} N^{2-\beta} \leq N^2$ .

We have  $1 \lesssim N_3 \leq N^{\frac{\beta}{\beta+1}}$ . We get from (3.4) and (3.20) that

$$\begin{aligned}
(3.20) &\lesssim \sup_{N \gtrsim 1} \sum_{N_3 \leq N^{\frac{\beta}{\beta+1}}, N_3 \lesssim 1} \sum_{L_3 \sim N^2 N_3, L_{med} \geq N^2} \frac{N_3 N^{2\rho}}{L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} L_3^{\frac{1}{2}-\delta}} L_{min}^{\frac{1}{2}} N_3^{\frac{1}{2}} \\
&\lesssim \sup_{N \gtrsim 1} \sum_{N_3 \leq N^{\frac{\beta}{\beta+1}}} \sum_{L_3 \sim N^2 N_3, L_{med} \geq N^2} \frac{N_3^{1+\delta} N^{2\rho-1+2\delta}}{L_{med}^{\frac{1}{2}}} \\
&\lesssim \sup_{N \gtrsim 1} \sum_{N_3 \leq N^{\frac{\beta}{\beta+1}}} \sum_{L_{min} \leq L_{med}} \frac{N_3^{1+\delta} N^{2\rho-2+4\delta}}{L_{med}^{\delta}} \\
&\lesssim \sup_{N \gtrsim 1} \sum_{N_3 \leq N^{\frac{\beta}{\beta+1}}} N_3^{1+\delta} N^{2\rho-2+4\delta} \\
&\lesssim \sup_{N \gtrsim 1} N^{\frac{\beta}{\beta+1}(1+\delta)+2\rho-2+4\delta} \lesssim 1
\end{aligned} \tag{3.29}$$

for  $\delta > 0$  and  $\beta > 0$  small, since the inequality  $0 \leq \rho < 1$  implies

$$\frac{\beta}{\beta+1}(1+\delta)+2\rho-2+4\delta < 0$$

for  $\delta > 0$  and  $\beta > 0$  small. We complete the estimate in Case A.

**The estimate in Case B.** In this case, we have  $L_{max} \sim N^2 N_3$  and  $L_{med} \gtrsim N^{2\alpha}$ , and then  $N_3^{\frac{1}{2}} \ll N^{\frac{1}{2}} \leq L_{med}^{\frac{1}{2\alpha}}$ .

When  $L_{med} \geq N_3^{\beta+1} N^{2-\beta} \geq N^{2\alpha}$ , we have  $N_3 \geq N^{\frac{2\alpha-2+\beta}{\beta+1}}$ . By using  $N_3 \lesssim 1$  we get from (3.4) and (3.20) that

$$\begin{aligned}
(3.20) &\lesssim \sup_{N \gtrsim 1} \sum_{N_1 \sim N_2 \sim N, N_3 \lesssim 1} \sum_{L_{max} \sim N^2 N_3, L_{med} \geq N_3^{\beta+1} N^{2-\beta}} \frac{N_3 N^{2\rho}}{L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} L_3^{\frac{1}{2}-\delta}} L_{min}^{\frac{1}{2}} N_3^{\frac{1}{2}} \\
&\lesssim \sup_{N \gtrsim 1} \sum_{N_3 \lesssim 1} \sum_{L_{max} \sim N^2 N_3, L_{med} \geq N_3^{\beta+1} N^{2-\beta}} \frac{N_3^{1+\delta} N^{2\rho-1+2\delta}}{L_{med}^{\frac{1}{2}}} \\
&\lesssim \sup_{N \gtrsim 1} \sum_{N_3 \lesssim 1} \sum_{L_{med} \lesssim L_{max} \sim N^2 N_3} \frac{N_3^{\frac{1}{2}-\frac{\beta}{2}+\delta(2+\beta)} N^{2\rho-2+\frac{\beta}{2}+\delta(4-\beta)}}{L_{med}^{\delta}} \\
&\lesssim \sup_{N \gtrsim 1} \sum_{N_3 \lesssim 1} N_3^{\frac{1}{2}-\frac{\beta}{2}+\delta(2+\beta)} N^{2\rho-2+\frac{\beta}{2}+\delta(4-\beta)} \\
&\lesssim \sup_{N \gtrsim 1} N^{2\rho-2+\frac{\beta}{2}+\delta(4-\beta)} \lesssim 1,
\end{aligned} \tag{3.30}$$

for  $\delta > 0$  and  $\beta > 0$  small, since the inequality  $\frac{1}{2} < \rho < 1$  implies

$$2\rho-2+\frac{\beta}{2}+\delta(4-\beta) < 0, \quad \frac{1}{2}-\frac{\beta}{2}+\delta(2+\beta) > 0$$

for  $\delta > 0$  and  $\beta > 0$  small.

When  $L_{med} \geq N_3^{\beta+1} N^{2-\beta}$ ,  $L_{med} \gtrsim N^{2\alpha}$  and  $N_3^{\beta+1} N^{2-\beta} \leq N^{2\alpha}$ , we have  $N_3 \leq N^{\frac{2\alpha-2+\beta}{\beta+1}}$ . We

get from (3.4) and (3.20) that

$$\begin{aligned}
(3.20) &\lesssim \sup_{N \gtrsim 1} \sum_{N_3 \leq N^{\frac{2\alpha-2+\beta}{\beta+1}}, N_3 \lesssim 1} \sum_{L_3 \sim N^2 N_3, L_{med} \geq N^{2\alpha}} \frac{N_3 N^{2\rho}}{L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} L_3^{\frac{1}{2}-\delta}} L_{min}^{\frac{1}{2}} N_3^{\frac{1}{2}} \\
&\lesssim \sup_{N \gtrsim 1} \sum_{N_3 \leq N^{\frac{2\alpha-2+\beta}{\beta+1}}} \sum_{L_3 \sim N^2 N_3, L_{med} \geq N^{2\alpha}} \frac{N_3^{1+\delta} N^{2\rho-1+2\delta}}{L_{med}^{\frac{1}{2}}} \\
&\lesssim \sup_{N \gtrsim 1} \sum_{N_3 \leq N^{\frac{2\alpha-2+\beta}{\beta+1}}} \sum_{L_{min} \leq L_{med}} \frac{N_3^{1+\delta} N^{2\rho-1+2\delta-(1-2\delta)\alpha}}{L_{med}^{\delta}} \\
&\lesssim \sup_{N \gtrsim 1} \sum_{N_3 \leq N^{\frac{2\alpha-2+\beta}{\beta+1}}} N_3^{1+\delta} N^{2\rho-1+2\delta-(1-2\delta)\alpha} \\
&\lesssim \sup_{N \gtrsim 1} N^{\frac{2\alpha-2+\beta}{\beta+1}(1+\delta)+2\rho-1+2\delta-(1-2\delta)\alpha} \lesssim 1
\end{aligned} \tag{3.31}$$

for  $\delta > 0$  and  $\beta > 0$  small, since the inequality  $\frac{1}{2} < \rho < 1$  implies

$$\frac{2\alpha-2+\beta}{\beta+1}(1+\delta)+2\rho-1+2\delta-(1-2\delta)\alpha < 0$$

for  $\delta > 0$  and  $\beta > 0$  small.

When  $L_{med} \leq N_3^{\beta+1} N^{2-\beta}$  and  $L_{med} \gtrsim N^{2\alpha}$ , we have  $1 \gtrsim N_3 \geq N^{\frac{2\alpha-2+\beta}{\beta+1}}$ . We get from (3.4) and (3.20) that

$$\begin{aligned}
(3.20) &\lesssim \sup_{N \gtrsim 1} \sum_{N_3 \gtrsim 1} \sum_{L_{med} \leq N_3^{\beta+1} N^{2-\beta}, L_3 \sim N^2 N_3} \frac{N_3 N^{2\rho}}{L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} L_3^{\frac{1}{2}-\delta}} L_{min}^{\frac{1}{2}} N_3^{-\frac{1}{2\beta}} N^{\frac{\beta-2}{2\beta}} L_{med}^{\frac{1}{2\beta}} \\
&\lesssim \sup_{N \gtrsim 1} \sum_{N_3 \gtrsim 1} \sum_{L_{max} \sim N^2 N_3} \frac{N_3^{\frac{1}{2}+(2+\beta)\delta-\frac{\beta}{2}} N^{2\rho-2+2\delta+\frac{\beta}{2}+(2-\beta)\delta}}{L_{med}^{\delta}} \\
&\lesssim \sup_{N \gtrsim 1} \sum_{N_3 \gtrsim 1} N_3^{\frac{1}{2}+(2+\beta)\delta-\frac{\beta}{2}} N^{2\rho-2+2\delta+\frac{\beta}{2}+(2-\beta)\delta} \\
&\lesssim \sup_{N \gtrsim 1} N^{2\rho-2+2\delta+\frac{\beta}{2}+(2-\beta)\delta} \lesssim 1
\end{aligned} \tag{3.32}$$

for  $\delta > 0$  and  $\beta > 0$  small, since the inequality  $\frac{1}{2} < \rho < 1$  means

$$2\rho-2+2\delta+\frac{\beta}{2}+(2-\beta)\delta < 0, \quad \frac{1}{2}+(2+\beta)\delta-\frac{\beta}{2} > 0$$

for  $\delta > 0$  and  $\beta > 0$  small. We complete the estimate in Case B.

**The estimate in Case C.** In this case, we have  $L_{max} \sim N^2 N_2$  and  $L_{med} \gtrsim N^{2\alpha}$ , and then  $N_2^{\frac{1}{2}} \leq N^{\frac{1}{2}} \leq L_{med}^{\frac{1}{2\alpha}}$ . We get from (3.4) and (3.20) that

$$\begin{aligned}
(3.20) &\lesssim \sup_{N \gtrsim 1} \sum_{N \sim N_1 \sim N_3 \gg N_2 \gtrsim 1} \sum_{L_1, L_3 \leq L_2 \sim N^2 N_2} \frac{N_3^{1-\rho} N_1^\rho N_2^\rho}{L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} L_3^{\frac{1}{2}-\delta}} \|X_{N_1, N_2, N_3; L_{max}; L_1, L_2, L_3}\|_{[3, \mathbb{R} \times \mathbb{R}]} \\
&\lesssim \sup_{N \gtrsim 1} \sum_{N \sim N_1 \sim N_3 \gg N_2 \gtrsim 1} \sum_{L_1, L_3 \leq L_2 \sim N^2 N_2} \frac{N_3^\rho N_1^\rho N_2^{1-\rho}}{L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} L_3^{\frac{1}{2}}} \|X_{N_1, N_2, N_3; L_{max}; L_1, L_2, L_3}\|_{[3, \mathbb{R} \times \mathbb{R}]} .
\end{aligned}$$

By symmetry and the estimate obtained in Case A we get

$$(3.20) \lesssim \sup_{N \gtrsim 1} \sum_{N \sim N_1 \sim N_3 \gg N_2} \sum_{L_1, L_3 \leq L_2 \sim N^2 N_2} \frac{N_3^{1-\rho} N_1^\rho N_2^\rho}{L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} L_3^{\frac{1}{2}-\delta}} \|X_{N_1, N_2, N_3; H; L_1, L_2, L_3}\|_{[3, R \times R]} \lesssim 1. \quad (3.33)$$

We complete the estimate in Case C.

**The estimate in Case D.** In this case, we have  $L_{max} = L_2 \sim N^2 N_2$  and  $L_{med} \gtrsim N^{2\alpha}$ , and then  $\alpha < 1$ . We get from (3.4) and (3.20) that

$$\begin{aligned} (3.20) &\lesssim \sup_{N \gtrsim 1} \sum_{N_1 \sim N_3 \sim N, N_2 \lesssim 1} \sum_{L_{max} \sim N^2 N_2, L_{med} \geq N^{2\alpha}} \frac{N}{L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} L_3^{\frac{1}{2}-\delta}} L_{min}^{\frac{1}{2}} N_2^{\frac{1}{2}} \\ &\lesssim \sup_{N \gtrsim 1} \sum_{N_1 \sim N_3 \sim N, N_2 \lesssim 1} \sum_{L_{max} \sim N^2 N_2, L_{med} \geq N^{2\alpha}} \frac{N N_2^{\frac{1}{2}}}{L_{max}^{\frac{1}{2}} L_{med}^{\frac{1}{2}-\delta}} \\ &\lesssim \sup_{N \gtrsim 1} \sum_{N_1 \sim N_3 \sim N, N_2 \lesssim 1} \sum_{L_{max} \sim N^2 N_2, L_{med} \geq N^{2\alpha}} \frac{L_{max}^\delta}{L_{med}^{\frac{1}{2}}} \\ &\lesssim \sup_{N \gtrsim 1} \sum_{N_1 \sim N_3 \sim N, N_2 \lesssim 1} \sum_{L_{max} \sim N^2 N_2, L_{med} \geq N^{2\alpha}} \frac{N_2^\delta N^{-\alpha+2\delta(1+\alpha)}}{L_{med}^\delta} \\ &\lesssim \sup_{N \gtrsim 1} \sum_{N_2 \lesssim 1} N_2^\delta N^{-\alpha+2\delta(1+\alpha)} \lesssim \sup_{N \gtrsim 1} N^{-\alpha+2\delta(1+\alpha)} \lesssim 1, \end{aligned} \quad (3.34)$$

for  $\delta > 0$  small, since we have  $0 \leq \alpha < 1$  in this case. We complete the estimate where the case (2b) in Lemma 3.1 applies.

To finish the estimate of (3.20) it remains to deal with the case where (2C) in Lemma 3.1 holds. When  $N_{min} = N_3 \gtrsim 1$ , we have  $L_3 \ll L_{max}$  and  $N^2 N_3 \geq N^{2\alpha}$ , and then

$$\begin{aligned} (3.20) &\lesssim \sup_{N \gtrsim 1} \sum_{N_{med} \sim N_{max} \sim N, N_{min} \gtrsim 1} \sum_{L_3 \ll L_{max} \sim N^2 N_{min}} \frac{N_{min}^{1-\rho} N^{2\rho}}{L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} L_3^{\frac{1}{2}-\delta}} L_{min}^{\frac{1}{2}} N^{-1} L_{med}^{\frac{1}{2}} \\ &\lesssim \sup_{N \gtrsim 1} \sum_{N_{med} \sim N_{max} \sim N, N_{min} \gtrsim 1} \sum_{L_{max} \sim N^2 N_{min}} \frac{N_{min}^{1-\rho} N^{2\rho-1}}{L_{max}^{\frac{1}{2}} L_{min}^{\frac{1}{2}} L_{med}^{\frac{1}{2}-\delta}} L_{min}^{\frac{1}{2}} N^{-1} L_{med}^{\frac{1}{2}} \\ &\lesssim \sup_{N \gtrsim 1} \sum_{N_{med} \sim N_{max} \sim N, N_{min} \gtrsim 1} \sum_{L_{max} \sim N^2 N_{min}} \frac{N_{min}^{1-\rho} N^{2\rho}}{L_{max}^{\frac{1}{2}} L_{max}^{\frac{1}{2}-\delta}} \\ &\lesssim \sup_{N \gtrsim 1} \sum_{N_{med} \sim N_{max} \sim N, N_{min} \gtrsim 1} \sum_{L_{max} \sim N^2 N_{min}} \frac{N_{min}^{\frac{1}{2}-\rho+2\delta} N^{2\rho-2+4\delta}}{L_{max}^\delta} \\ &\lesssim \sup_{N \gtrsim 1} \sum_{N_{med} \sim N_{max} \sim N, N_{min} \gtrsim 1} N_{min}^{\frac{1}{2}-\rho+2\delta} N^{2\rho-2+4\delta} \\ &\lesssim \sup_{N \gtrsim 1} N^{2\rho-2+4\delta} \lesssim 1, \end{aligned} \quad (3.35)$$

for  $\delta > 0$  small.

When  $N_{min} = N_3 \lesssim 1$ , we have  $L_3 \ll L_{max}$ , and

$$\begin{aligned}
(3.20) &\lesssim \sup_{N \gtrsim 1} \sum_{N_{med} \sim N_{max} \sim N, N_{min} \lesssim 1} \sum_{L_3 \ll L_{max} \sim \max\{N^2 N_{min}, N^{2\alpha}\}} \frac{N_{min} N^{2\rho}}{L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} L_3^{\frac{1}{2}-\delta}} L_{min}^{\frac{1}{2}} N^{-1} L_{med}^{\frac{1}{2}} \\
&\lesssim \sup_{N \gtrsim 1} \sum_{N_{med} \sim N_{max} \sim N, N_{min} \lesssim 1} \sum_{L_{max} \sim \max\{N^2 N_{min}, N^{2\alpha}\}} \frac{N_{min} N^{2\rho-1}}{L_{max}^{\frac{1}{2}} L_{min}^{\frac{1}{2}} L_{med}^{\frac{1}{2}-\delta}} L_{min}^{\frac{1}{2}} N^{-1} L_{med}^{\frac{1}{2}} \\
&\lesssim \sup_{N \gtrsim 1} \sum_{N_{med} \sim N_{max} \sim N, N_{min} \lesssim 1} \sum_{L_{max} \sim \max\{N^2 N_{min}, N^{2\alpha}\}} \frac{N_{min} N^{2\rho}}{L_{max}^{\frac{1}{2}} L_{max}^{\frac{1}{2}-\delta}} \\
&\lesssim \sup_{N \gtrsim 1} \sum_{N_{med} \sim N_{max} \sim N, 1 \gtrsim N_{min} \geq N^{2\alpha-2}} \sum_{L_{max} \sim N^2 N_{min}} \frac{N_{min} N^{2\rho}}{L_{max}^{\frac{1}{2}} L_{max}^{\frac{1}{2}-\delta}} \\
&\quad + \sup_{N \gtrsim 1} \sum_{N_{med} \sim N_{max} \sim N, N_{min} \leq N^{2\alpha-2}} \sum_{L_{max} \sim N^{2\alpha}} \frac{N_{min} N^{2\rho}}{L_{max}^{\frac{1}{2}} L_{max}^{\frac{1}{2}-\delta}} \\
&\lesssim \sup_{N \gtrsim 1} \sum_{N_{med} \sim N_{max} \sim N, 1 \gtrsim N_{min} \geq N^{2\alpha-2}} \sum_{L_{max} \sim N^2 N_{min}} \frac{N_{min}^{\frac{1}{2}+2\delta} N^{2\rho-2+4\delta}}{L_{max}^\delta} \\
&\quad + \sup_{N \gtrsim 1} \sum_{N_{med} \sim N_{max} \sim N, N_{min} \leq N^{2\alpha-2}} \sum_{L_{max} \sim N^{2\alpha}} \frac{N_{min} N^{2\rho-1-\alpha+2\alpha\delta}}{L_{max}^\delta} \\
&\lesssim \sup_{N \gtrsim 1} \sum_{N_{med} \sim N_{max} \sim N, N_{min} \geq N^{2\alpha-2}} N_{min}^{\frac{1}{2}-\rho+2\delta} N^{2\rho-2+4\delta} \\
&\quad + \sup_{N \gtrsim 1} \sum_{N_{med} \sim N_{max} \sim N, N_{min} \leq N^{2\alpha-2}} N_{min} N^{2\rho-1-\alpha+2\alpha\delta} \\
&\lesssim \sup_{N \gtrsim 1} N^{2\rho-2+4\delta} + \sup_{N \gtrsim 1} N^{2\rho-3+\alpha+2\alpha\delta} \lesssim 1, \tag{3.36}
\end{aligned}$$

for  $\delta > 0$  small, since the inequalities  $\frac{1}{2} < \rho < 1$  and  $0 \leq \alpha \leq 1$  imply

$$2\rho - 3 + \alpha + 2\alpha\delta < 0$$

for  $\delta > 0$  small.

When  $N_{min} = N_2 \gtrsim 1$ , we have  $L_2 \ll L_{max}$  and  $N^2 N_{min} \gtrsim N^{2\alpha}$ , and then

$$\begin{aligned}
(3.20) &\lesssim \sup_{N \gtrsim 1} \sum_{N_{med} \sim N_{max} \sim N, N_{min} \gtrsim 1} \sum_{L_2 \ll L_{max} \sim N^2 N_{min}} \frac{N_{min}^\rho N}{L_{min}^{\frac{1}{2}} L_{med}^{\frac{1}{2}} L_{max}^{\frac{1}{2}-\delta}} L_{min}^{\frac{1}{2}} N^{-1} L_{med}^{\frac{1}{2}} \\
&\lesssim \sup_{N \gtrsim 1} \sum_{N_{med} \sim N_{max} \sim N, N_{min} \gtrsim 1} \sum_{L_2 \ll L_{max} \sim N^2 N_{min}} \frac{N_{min}^\rho}{L_{max}^{\frac{1}{2}-\delta}} \\
&\lesssim \sup_{N \gtrsim 1} \sum_{N_{med} \sim N_{max} \sim N, N_{min} \gtrsim 1} \sum_{L_{max} \sim N^2 N_{min}} \frac{N_{min}^{\rho-\frac{1}{2}+2\delta} N^{-1+2\delta}}{L_{max}^\delta} \\
&\lesssim \sup_{N \gtrsim 1} \sum_{N_{med} \sim N_{max} \sim N, N_{min} \gtrsim 1} N_{min}^{\rho-\frac{1}{2}+2\delta} N^{-1+2\delta} \\
&\lesssim \sup_{N \gtrsim 1} N^{-1+2\delta} \lesssim 1, \tag{3.37}
\end{aligned}$$

for  $\delta > 0$  small.



When  $N_{\min} = N_2 \lesssim 1$ , we have  $L_2 \ll L_{\max}$ . Note that  $N^{-1}L_{\text{med}}^{\frac{1}{2}} \leq N_2^{\frac{1}{2}}$  implies  $L_{\text{med}} \leq N^2 N_2$ . We get from (3.5) and (3.20) that

$$\begin{aligned}
(3.20) &\lesssim \sup_{N \gtrsim 1} \sum_{N_1 \sim N_3 \sim N, N_2 \lesssim 1} \sum_{L_2 \ll L_{\max} \sim \max\{N^2 N_2, N^{2\alpha}\}} \frac{N}{L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} L_3^{\frac{1}{2}-\delta}} L_2^{\frac{1}{2}} \min\{N^{-1}L_{\text{med}}^{\frac{1}{2}}, N_2^{\frac{1}{2}}\} \\
&\lesssim \sup_{N \gtrsim 1} \sum_{N_1 \sim N_3 \sim N, N_2 \lesssim 1} \sum_{L_2 \ll L_{\max} \sim \max\{N^2 N_2, N^{2\alpha}\}, L_{\text{med}} \leq N^2 N_2} \frac{N}{L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} L_3^{\frac{1}{2}-\delta}} L_2^{\frac{1}{2}} N^{-1} L_{\text{med}}^{\frac{1}{2}} \\
&\quad + \sup_{N \gtrsim 1} \sum_{N_1 \sim N_3 \sim N, N_2 \lesssim 1} \sum_{L_2 \ll L_{\max} \sim \max\{N^2 N_2, N^{2\alpha}\}, L_{\text{med}} \geq N^2 N_2} \frac{N}{L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} L_3^{\frac{1}{2}-\delta}} L_2^{\frac{1}{2}} N_2^{\frac{1}{2}} \\
&\lesssim \sup_{N \gtrsim 1} \sum_{N_1 \sim N_3 \sim N, N_2 \lesssim 1} \sum_{L_2 \ll L_{\max} \sim \max\{N^2 N_2, N^{2\alpha}\}, L_{\text{med}} \leq N^2 N_2} \frac{N^{2\delta} N_2^{\delta}}{L_{\max}^{\frac{1}{2}-\delta} L_{\text{med}}^{\delta}} \\
&\quad + \sup_{N \gtrsim 1} \sum_{N_1 \sim N_3 \sim N, N_2 \lesssim 1} \sum_{L_2 \ll L_{\max} \sim \max\{N^2 N_2, N^{2\alpha}\}, L_{\text{med}} \geq N^2 N_2} \frac{N N_2^{\frac{1}{2}}}{L_{\max}^{\frac{1}{2}-\delta} L_{\text{med}}^{\delta}} \\
&\lesssim \sup_{N \gtrsim 1} \sum_{N_2 \geq N^{2\alpha-2}} \sum_{L_{\max} \sim N^2 N_2, L_{\text{med}} \leq N^2 N_2} \frac{N^{2\delta} N_2^{\delta}}{L_{\max}^{\frac{1}{2}-\delta} L_{\text{med}}^{\delta}} \\
&\quad + \sup_{N \gtrsim 1} \sum_{N_2 \leq N^{2\alpha-2}} \sum_{L_{\max} \sim N^{2\alpha}, L_{\text{med}} \leq N^2 N_2} \frac{N^{2\delta} N_2^{\delta}}{L_{\max}^{\frac{1}{2}-\delta} L_{\text{med}}^{\delta}} \\
&\quad + \sup_{N \gtrsim 1} \sum_{N_1 \sim N_3 \sim N, N_2 \lesssim 1} \sum_{L_2 \ll L_{\max} \sim \max\{N^2 N_2, N^{2\alpha}\}, L_{\text{med}} \geq N^2 N_2} \frac{N N_2^{\frac{1}{2}}}{L_{\max}^{\frac{1}{2}-\delta} L_{\text{med}}^{\delta}} \\
&\lesssim \sup_{N \gtrsim 1} \sum_{N_2 \geq N^{2\alpha-2}} N^{4\delta-1} N_2^{2\delta-\frac{1}{2}} + \sup_{N \gtrsim 1} \sum_{N_2 \leq N^{2\alpha-2}} N^{-\alpha+2\delta(\alpha+1)} N_2^{\delta} \\
&\quad + \sup_{N \gtrsim 1} \sum_{N_2 \lesssim N^{2\alpha-2}} \sum_{L_{\max} \sim N^{2\alpha}, L_{\text{med}} \geq N^2 N_2} \frac{N^{2\delta} N_2^{\delta}}{L_{\max}^{\frac{1}{2}-\delta} L_{\text{med}}^{\delta}} \\
&\lesssim \sup_{N \gtrsim 1} N^{-\alpha+4\alpha\delta} + \sup_{N \gtrsim 1} \sum_{N_2 \lesssim N^{2\alpha-2}} N^{-\alpha+2\delta(1+\alpha)} N_2^{\delta} \\
&\lesssim \sup_{N \gtrsim 1} N^{-\alpha+4\alpha\delta} \lesssim 1, \tag{3.38}
\end{aligned}$$

for  $\delta > 0$  small. By symmetry, the same estimate holds when  $N_{\min} = N_3$ . We complete the proof of (3.20)  $\lesssim 1$ .  $\square$

**Theorem 3.1** *Given  $s \in (-\min\{\frac{3+2\alpha}{4}, 1\}, -\frac{1}{2})$ , there exists  $\mu > 0$ ,  $\delta > 0$  such that for any  $u, v \in X_{\alpha}^{\frac{1}{2}, s}$  with compact support in  $[-T, T]$ ,*

$$\|\partial_x(uv)\|_{X_{\alpha}^{-\frac{1}{2}+\delta, s}} \lesssim T^{\mu} \|u\|_{X_{\alpha}^{\frac{1}{2}, s}} \|v\|_{X_{\alpha}^{\frac{1}{2}, s}}. \tag{3.39}$$

*Proof.* By duality, (3.39) is equivalent to, for all  $w \in X_{\alpha}^{\frac{1}{2}-\delta, s}$ ,

$$|\langle \partial_x(uv), w \rangle| \lesssim T^{\mu} \|u\|_{X_{\alpha}^{\frac{1}{2}, s}} \|v\|_{X_{\alpha}^{\frac{1}{2}, s}} \|w\|_{X_{\alpha}^{\frac{1}{2}, s-\delta}}. \tag{3.40}$$

Then the theorem follows from Lemma 4 in [7], (3.40) and Lemma 3.2.  $\square$

The following theorem is a direct consequence of Theorem 3.1 together with the triangle inequality

$$\langle \xi \rangle^s \leq \langle \xi \rangle^{s_c} \langle \xi_1 \rangle^{s-s_c} + \langle \xi \rangle^{s_c} \langle \xi - \xi_1 \rangle^{s-s_c}, \forall s \geq s_c.$$

**Theorem 3.2** Given  $s_c \in (-\min\{\frac{3+2\alpha}{4}, 1\}, -\frac{1}{2})$ , there exists  $\mu > 0$ ,  $\delta > 0$  such that for any  $s \geq s_c$  and for any couple  $(u, v) \in X_\alpha^{\frac{1}{2}, s}$  with compact support in  $[-T, T]$ ,

$$\|\partial_x(uv)\|_{X_\alpha^{-\frac{1}{2}+\delta, s}} \lesssim T^\mu \left( \|u\|_{X_\alpha^{\frac{1}{2}, s_c}} \|v\|_{X_\alpha^{\frac{1}{2}, s}} + \|u\|_{X_\alpha^{\frac{1}{2}, s}} \|v\|_{X_\alpha^{\frac{1}{2}, s_c}} \right). \quad (3.41)$$

**The proof of Theorem 1.1.** The proof is similar to that of Theorem 1 in [7], we omit it.

## 4 Ill-posedness results

In this section we give some ill-posedness results.

**Theorem 4.1** Let  $\frac{1}{2} \leq \alpha \leq 1$ ,  $s < -1$  and  $T > 0$ . Then there does not exist a space  $Y_T$  continuously embedded in  $C([0, T], H^s(\mathbb{R}))$  such that

$$\|W(t)\varphi\|_{Y_T} \lesssim \|\varphi\|_{H^s}, \quad \forall \varphi \in H^s(\mathbb{R}), \quad (4.1)$$

$$\left\| \int_0^t W(t-t') \partial_x[u^2(t')] dt' \right\|_{Y_T} \lesssim \|u\|_{Y_T}^2, \quad \forall u \in Y_T. \quad (4.2)$$

*Proof.* Suppose that there exists a space  $Y_T$  such that (4.1) and (4.2) hold. For any  $t \in [0, T]$ , taking  $u = W(t)\varphi$  and since  $Y_T$  is continuously embedded in  $C([0, T], H^s(\mathbb{R}))$ , we get

$$\left\| \int_0^t W(t-t') \partial_x[(W(t')\varphi)^2] dt' \right\|_{H^s} \lesssim \left\| \int_0^t W(t-t') \partial_x[(W(t')\varphi)^2] dt' \right\|_{Y_T} \lesssim \|\varphi\|_{H^s}^2. \quad (4.3)$$

We show now that (4.3) fails by choosing an appropriate sequence  $\{\varphi_N\}$ . Let  $\{\varphi_N\}$  be the real-valued function defined through its Fourier transform by

$$\hat{\varphi}_N = N^{-s}[\chi_{I_N}(\xi) + \chi_{-I_N}(\xi)],$$

where  $I_N = [N, N+2]$ , so  $\varphi_N \in \mathcal{S}$ . Note that  $\|\varphi\|_{H^s} \sim 1$ , setting

$$u_{1,N}(t, x) = W(t)\varphi_N, \quad u_{2,N}(t, x) = \int_0^t W(t-t') \partial_x[(W(t')\varphi)^2] dt'.$$

and taking  $x$ -Fourier transform, we will get

$$\mathcal{F}_x(u_{2,N}(t, \cdot))(\xi) = \int_0^t e^{-(t-t')|\xi|^{2\alpha}} e^{i(t-t')\xi^3} (i\xi) [\mathcal{F}_x(u_{1,N}(t')) * \mathcal{F}_x(u_{1,N}(t'))](\xi) dt',$$

where

$$\begin{aligned} & [\mathcal{F}_x(u_{1,N}(t')) * \mathcal{F}_x(u_{1,N}(t'))](\xi) = [\mathcal{F}_x(W(t')\varphi_N) * \mathcal{F}_x(W(t')\varphi_N)](\xi) \\ &= \int_{\mathbb{R}} \hat{\varphi}_N(\xi_1) \hat{\varphi}_N(\xi - \xi_1) e^{-(|\xi_1|^{2\alpha} + |\xi - \xi_1|^{2\alpha})t'} e^{i(\xi_1^3 + (\xi - \xi_1)^3)t'} d\xi_1. \end{aligned}$$

Hence

$$\begin{aligned} \mathcal{F}_x(u_{2,N}(t, \cdot))(\xi) &= e^{-t|\xi|^{2\alpha}} e^{it\xi^3} (i\xi) \int_{\mathbb{R}} \hat{\varphi}_N(\xi_1) \hat{\varphi}_N(\xi - \xi_1) \\ &\quad \times \frac{e^{-(|\xi_1|^{2\alpha} + |\xi - \xi_1|^{2\alpha} - |\xi|^{2\alpha})t} e^{i(\xi_1^3 + (\xi - \xi_1)^3 - \xi^3)t} - 1}{-(|\xi_1|^{2\alpha} + |\xi - \xi_1|^{2\alpha} - |\xi|^{2\alpha}) + i(\xi_1^3 + (\xi - \xi_1)^3 - \xi^3)} d\xi_1, \end{aligned}$$

$$\begin{aligned}
& \| u_{2,N}(t) \|_{H^s}^2 \geq \int_{-\frac{1}{2}}^{\frac{1}{2}} < \xi >^{2s} | \mathcal{F}_x(u_{2,N}(t, \cdot))(\xi) |^2 d\xi \\
& = N^{-4s} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \int_{K_\xi} \frac{e^{-(|\xi_1|^{2\alpha} + |\xi - \xi_1|^{2\alpha})t} e^{i(\xi_1^3 + (\xi - \xi_1)^3 - \xi^3)t} - e^{-|\xi|^{2\alpha}t}}{-(|\xi_1|^{2\alpha} + |\xi - \xi_1|^{2\alpha} - |\xi|^{2\alpha}) + i(\xi_1^3 + (\xi - \xi_1)^3 - \xi^3)} d\xi_1 \right|^2 \\
& \quad \times < \xi >^{2s} | \xi |^2 d\xi,
\end{aligned}$$

where

$$K_\xi = \{\xi_1 \mid \xi - \xi_1 \in I_N, \xi_1 \in -I_N\} \cup \{\xi_1 \mid \xi - \xi_1 \in -I_N, \xi_1 \in I_N\}.$$

Note that for any  $\xi \in [-\frac{1}{2}, \frac{1}{2}]$ . One has  $\text{mes}(K_\xi) \gtrsim 1$  and

$$|\xi_1|^{2\alpha} + |\xi - \xi_1|^{2\alpha} - |\xi|^{2\alpha} \sim N^{2\alpha}, \quad \xi_1^3 + (\xi - \xi_1)^3 - \xi^3 = 3\xi\xi_1(\xi - \xi_1) \sim N^2.$$

We have

$$e^{-|\xi|^{2\alpha}t} - \text{Re}(e^{-(|\xi_1|^{2\alpha} + |\xi - \xi_1|^{2\alpha})t} e^{i(\xi_1^3 + (\xi - \xi_1)^3 - \xi^3)t}) \geq e^{-(1/2)^{2\alpha}t} - e^{-2(N+2)^{2\alpha}t},$$

which leads to

$$\left| \int_{K_\xi} \frac{e^{-(|\xi_1|^{2\alpha} + |\xi - \xi_1|^{2\alpha})t} e^{i(\xi_1^3 + (\xi - \xi_1)^3 - \xi^3)t} - e^{-|\xi|^{2\alpha}t}}{-(|\xi_1|^{2\alpha} + |\xi - \xi_1|^{2\alpha} - |\xi|^{2\alpha}) + i(\xi_1^3 + (\xi - \xi_1)^3 - \xi^3)} d\xi_1 \right| \geq \frac{e^{-(1/2)^{2\alpha}t} - e^{-2(N+2)^{2\alpha}t}}{N^{2\alpha} + N^2}.$$

Thus

$$\| u_{2,N}(t) \|_{H^s}^2 \geq N^{-4s} \left( \frac{e^{-(1/2)^{2\alpha}t} - e^{-2(N+2)^{2\alpha}t}}{N^{2\alpha} + N^2} \right)^2 \geq N^{-4s-4} \left( e^{-(1/2)^{2\alpha}t} - e^{-2(N+2)^{2\alpha}t} \right)^2. \quad (4.4)$$

(4.4) contradicts (4.3) when  $N$  is large enough.  $\square$

**Theorem 4.2** *Let  $\frac{1}{2} \leq \alpha \leq 1$  and  $s < -1$ . Then there does not exists any  $T$  such that (1.1) admits a unique local solution defined on the interval  $[0, T]$  and such that the flow-map*

$$\varphi \mapsto u(t), \quad t \in [0, T]$$

*is  $C^2$  differentiable at zero from  $H^s(\mathbb{R})$  to  $C([0, T]; H^s(\mathbb{R}))$ .*

*Proof.* Let  $u$  be a solution of (1.1). Then we have

$$u(t, x, \varphi) = W(t)\phi - \frac{1}{2} \int_0^t W(t-t') \partial_x(u(t', x, \phi)^2) dt'.$$

Assume now that the flow-map is  $C^2$ . Since  $u(t, x, 0) \equiv 0$ , we have

$$u_1(t, x) := \frac{\partial u}{\partial \phi}(t, x, 0)[h] = W(t)h,$$

$$\begin{aligned}
u_2(t, x) : &= \frac{\partial^2 u}{\partial^2 \phi}(t, x, 0)[h, h] = \int_0^t W(t-t') \partial_x(u_1(t', x))^2 dt' \\
&= \int_0^t W(t-t') \partial_x(W(t')h)^2 dt'.
\end{aligned}$$

Since the flow-map is  $C^2$  one must have

$$\| u_2(t) \|_{H^s} \leq \| h \|_{H^s}^2, \quad \forall h \in H^s(\mathbb{R}).$$

But this is exactly the estimate which has been shown to fail in the proof of Theorem 4.1.  $\square$

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